The basic invariants and high-n squeezed states of the time-dependent harmonic oscillator

1991 J. Phys. A: Math. Gen. 24 L1007

(http://iopscience.iop.org/0305-4470/24/17/008)

J. Phys. A: Math. Gen. 24 (1991) L1007–L1011. Printed in the UK

LETTER TO THE EDITOR

The basic invariants and high-n squeezed states of the time-dependent harmonic oscillator*

Tie-Zheng Qian†, Jing-Bo Xu† and Xiao-Chun Gao†‡§

† Department of Physics, Zhejiang University, Hangzhou 310027, People's Republic of China

‡ CCAST (World Laboratory), PO Box 8730, Beijing, People's Republic of China

§ Institute of theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China

Received 3 April 1991

Abstract. A new concept of the basic invariants is introduced. With the help of the basic invariants, new high-n squeezed states of the time-dependent harmonic oscillator are constructed. From the basic invariants, we also find a general form of invariants which has not been seen thus far in the literature.

The coherent states [1] and squeezed states [2] of the harmonic oscillator have long been of interest for their remarkable properties. For the time-dependent harmonic oscillator (THO) with the Hamiltonian

$$H(t)=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2 \tag{1}$$

Hartley and Ray [3] constructed the coherent states by making use of the Lewis–Riesenfeld [4] quantum invariant theory. Recently, Pedrosa [5] pointed out that these states are equivalent to the well known squeezed states which are also constructed by other authors [6, 7] with the time-evolution operator technique. In this letter, we introduce a new concept of the basic invariants. With the help of the basic invariants, not only the above-stated squeezed states can readily be obtained, but some new high-n squeezed states, which may be applied to the study of the thermal distribution for the systems, can also be constructed. From the basic invariants, we also find a general form of invariant which has not been seen thus far in the literature.

Lewis–Riesenfeld invariants can be used to study coherent states [3, 8]. Now, we introduce a new concept of the basic invariants, which are different from Lewis–Riesenfeld invariants. From the definition of the invariant

$$\mathrm{d}I(t)/\mathrm{d}t\equiv\partial I(t)/\partial t-\mathrm{i}[I(t),H(t)]=0 \tag{2}$$

we can find the formal solution

$$I(t)=U(t)I(0)U^{\dagger}(t) \tag{3}$$

where $U(t)$ is the time-evolution operator for the system with Hamiltonian $H(t)$. Although, in general, the formal solution is not convenient for actual calculations, it can help us to introduce a new concept of the basic invariants. For a one-dimensional

* This project is supported by Zhejiang Provincial Natural Science Foundation of China.

0305-4470/91/171007+05$03.50

system, there exist two basic invariants: $q(t)=U(t)qU^{\dagger}(t)$ and $p(t)=U(t)pU^{\dagger}(t)$ or two arbitrary linearly-independent combinations of them. We want to point out two facts.

(i) With $I(0)=q$ or p, two invariants $q(t)=U(t)qU^{\dagger}(t)$ and $p(t)=U(t)pU^{\dagger}(t)$ can be obtained by making use of equation (3). If I_1 and I_2 are both invariants, it is readily seen that I_1I_2 and $c_1I_1+c_2I_2$ (c_1 and c_2 are time-independent constants) are all invariants. With this in mind, a general invariant (not necessarily Hermitian) $I(t)=U(t)I(0)U^{\dagger}(t)$ ($I(0)=\sum_{m=0}^{M}\sum_{n=0}^{N}C_{mn}p^mq^n$ with C_{mn} being constant) can be expressed in terms of the power series in $q(t)$ and $p(t)$ as follows:

$$\begin{aligned}I(t)&=U(t)I(0)U^{\dagger}(t)\\&=U(t)\left(\sum_{m=0}^{M}\sum_{n=0}^{N}C_{mn}p^mq^n\right)U^{\dagger}(t)\\&=\sum_{m=0}^{M}\sum_{n=0}^{N}C_{mn}[U(t)pU^{\dagger}(t)]^m[U(t)qU^{\dagger}(t)]^n\\&=\sum_{m=0}^{M}\sum_{n=0}^{N}C_{mn}[p(t)]^m[q(t)]^n.\end{aligned}\tag{4}$$

It is in this sense that $q(t)$ and $p(t)$ are referred to as basic invariants. It is apparent that any two linearly-independent combinations of $q(t)$ and $p(t)$ can also be regarded as basic invariants.

(ii) If $|\Psi(t)\rangle_S$ is a solution of the Schrödinger equation for the system, then $I(t)|\Psi(t)\rangle_S$ is also a solution. This fact may be used to obtain a series of solutions from one solution of the Schrödinger equation by making use of the basic invariants. It is worth emphasizing that the basic invariants cannot be found in the literature. For example, the 'general form' of the invariant for the displaced harmonic oscillator was discussed by Xin Ma [9]. However, his form is not really general, for it fails to contain our basic invariants.

Since q, p and H constitute a quasi-algebra

$$[q,H]=\mathrm{i}\omega^2(t)p \qquad [p,H]=-\mathrm{i}q\tag{5}$$

it is not difficult to show that a general basic invariant for the system with Hamiltonian (1) is of the form

$$I_e(t)=\{x^{-1}(t)\cos[\theta(t)+\theta_0]+\dot{x}(t)\sin[\theta(t)+\theta_0]\}q-\{x(t)\sin[\theta(t)+\theta_0]\}p\tag{6}$$

with $\theta(t)=\int_0^t x^{-2}(t')\,\mathrm{d}t'$, where $x(t)$ is a c-number solution of the auxiliary equation

$$\ddot{x}+\omega^2(t)x=x^{-3}\tag{7}$$

and θ_0 is an initial phase angle. It is apparent that the initial conditions imposed on x, $\dot{x}$ can be arbitrarily chosen. With appropriate choices, two linearly-independent basic invariants can be obtained as follows:

$$I_b(t)=(1/2)^{1/2}\exp[\mathrm{i}\theta(t)]\{x^{-1}(t)q+\mathrm{i}[x(t)p-\dot{x}(t)q]\}\tag{8a}$$

$$I_b^{+}(t)=(1/2)^{1/2}\exp[-\mathrm{i}\theta(t)]\{x^{-1}(t)q-\mathrm{i}[x(t)p+\dot{x}(t)q]\}.\tag{8b}$$

with $[I_b(t),I_b^{\dagger}(t)]=1$. It is worthwhile to note that the existence of the factor $\exp[\mathrm{i}\theta(t)]$ in equation (8) guarantees the time-independent properties of the eigenvalues of $I_b(t)$ and reflects the fact that the basic invariants $I_b(t)$, $I_b^{\dagger}(t)$ are not Hermitian. In contrast, the invariants introduced thus far in the literature are all Hermitian (3, 4, 8, 9).

If at the initial time $t=0$ the eigenstate of $I_b(0)$ satisfies

$$I_b(0)|\Psi(0)\rangle = \xi|\Psi(0)\rangle \tag{9}$$

it is easily shown that

$$I_b(t)|\Psi(t)\rangle = \xi|\Psi(t)\rangle \tag{10}$$

where ξ is a complex constant. With the expression for $I_b(t)$, equation (10) becomes

$$D(z)S(r,\Phi)aS^\dagger(r,\Phi)D^\dagger(z)|\Psi(t)\rangle = 0 \tag{11}$$

where

$$a=(2\omega_0)^{-1/2}(\omega_0 q+\mathrm{i}p) \qquad D(z)=\exp[za^\dagger - z^* a] \tag{12}$$

$$S(r,\Phi)=\exp[\tfrac{1}{4}r\,\mathrm{e}^{\mathrm{i}\Phi}a^2 - \tfrac{1}{4}r\,\mathrm{e}^{-\mathrm{i}\Phi}a^{\dagger 2}] \tag{13}$$

with $\omega_0=1$ in the same units as ω in equation (1) and

$$\cosh(r/2)\exp(\mathrm{i}\alpha)=(x+x^{-1}-\mathrm{i}\dot{x})/2 \qquad \sinh(r/2)\exp(\mathrm{i}\beta)=(x-x^{-1}-\mathrm{i}\dot{x})/2 \tag{14}$$

$$\phi=\alpha-\beta \qquad z=\exp[-\mathrm{i}(\theta+\alpha)]\xi\cosh(r/2)-\exp[\mathrm{i}(\theta+\beta)]\xi^*\sinh(r/2). \tag{15}$$

Then, the state $|\Psi(t)\rangle$ can be expressed as

$$|\Psi(t)\rangle = \exp[\mathrm{i}\delta(t)]D(z)S(r,\phi)|0\rangle \tag{16}$$

where $|0\rangle$ is the vacuum state of $a^\dagger a$ and δ is found to be

$$\begin{aligned}\delta(t) &= \int_0^t \langle 0|S^\dagger D^\dagger[\mathrm{i}\partial/\partial t' - H(t')]DS|0\rangle\,\mathrm{d}t' \\ &= \tfrac{1}{4}\int_0^t \{2\dot{\phi}\sinh^2(r/2)-[(\omega^2+1)\cosh r-(\omega^2-1)\cos\phi\sinh r]\}\,\mathrm{d}t'.\end{aligned} \tag{17}$$

From (12), (13) and (16), the squeezed states $|\Psi(t)\rangle$ for the system are obtained.

Now, we discuss how the squeezing can be generated by tuning the oscillator frequency. To this end, we calculate the uncertainties $(\Delta q)^2$, $(\Delta p)^2$ for the squeezed state in (16):

$$(\Delta q)^2 = \langle\Psi(t)|q^2|\Psi(t)\rangle - (\langle\Psi(t)|q|\Psi(t)\rangle)^2 = \hbar x^2/2 \tag{18a}$$

$$(\Delta p)^2 = \langle\Psi(t)|p^2|\Psi(t)\rangle - (\langle\Psi(t)|p|\Psi(t)\rangle)^2 = \hbar(x^{-2}+\dot{x}^2)/2 \tag{18b}$$

$$(\Delta q)^2(\Delta p)^2 = \hbar^2(1+x^2\dot{x}^2)/4. \tag{18c}$$

In the adiabatic limit, from equation (7) we get

$$x(t)=[\omega(t)]^{-1/2} \qquad \dot{x}(t)=0 \tag{19a}$$

$$I_b(t)=[2\omega(t)]^{-1/2}\exp(\mathrm{i}\theta)(\omega(t)q+\mathrm{i}p]. \tag{19b}$$

From (18) and (19), it follows that the system will remain in the Glauber coherent state of mode $\omega(t)$ if it is initially in the Glauber coherent state of mode $\omega(0)$ in the adiabatic limit. This is to say that in the adiabatic cases it is impossible to generate squeezing. In order to generate squeezed states out of coherent states, it is necessary to have $\dot{\omega}(t)\neq 0$ which leads to $\dot{x}\neq 0$. This conclusion is in agreement with that obtained in [7, 10, 11].

We then proceed to construct the general form of the invariants by making use of the basic invariants $I_b(t)$ and $I_b^\dagger(t)$. The general form is

$$I(t)=\sum_{m=0}^{M}\sum_{n=0}^{N}A_{mn}[I_b^\dagger(t)]^m(I_b(t))^n \tag{20}$$

where M, N are positive integers and A_{mn} is a complex constant. It is worth pointing out that the Lewis–Riesenfeld invariant

$$\begin{aligned}I_c(t)&=\tfrac{1}{2}[q^2x^{-2}+(xp-\dot{x}q)^2]\\&=[I_b^\dagger(t)I_b(t)+\tfrac{1}{2}]\end{aligned} \tag{21}$$

can be readily obtained from the general form (20). In [3], Hartley and Ray constructed the coherent states of the time-dependent harmonic oscillator in terms of the eigenstates of the Lewis–Riesenfeld invariant $I_c(t)$. Here, we see clearly that the coherent states of the time-dependent harmonic oscillator are nothing but the eigenstates of the basic invariant $I_b(t)$.

Now, we turn to the discussion of new high-n squeezed states. Choosing appropriate coefficients A_{mn}, we obtain a special invariant

$$\begin{aligned}I_n(t)&=[I_b^\dagger(t)-\xi^*(t)]^n/(n!)\\&=\exp[-\mathrm{i}n(\theta+\alpha)]DS(a^\dagger)^nS^\dagger D^\dagger/(n!)^{1/2}.\end{aligned} \tag{22}$$

From (16), (17) and (22), we can construct the high-n squeezed states

$$\begin{aligned}|\Psi_n(t)\rangle&=I_n(t)|\Psi(t)\rangle\\&=\exp[-\mathrm{i}n(\theta+\alpha)+\mathrm{i}\delta]DS(a\dagger)^n(n!)^{-1/2}|0\rangle\\&=\exp\{\mathrm{i}[\delta-n(\theta+\alpha)\}DS|n\rangle\end{aligned} \tag{23}$$

where $|n\rangle$ is the eigenstate of $a^\dagger a$ with eigenvalue n. In order to see the squeezing effect, we calculate the uncertainties (Δq), (Δp) for $|\Psi(t)\rangle$ in (23)

$$(\Delta q)^2=(n+\tfrac{1}{2})\hbar x^2 \qquad (\Delta p)^2=(n+\tfrac{1}{2})\hbar(x^{-2}+\dot{x}^2) \tag{24a}$$

$$(\Delta q)^2(\Delta p)^2=(n+\tfrac{1}{2})^2\hbar^2(1+x^2\dot{x}^2). \tag{24b}$$

It is worth emphasizing that the high-n squeezed states constructed in this paper are the generalization of the high-n squeezed states of the harmonic oscillator with constant frequency in [12]. These generalized high-n squeezed states may be applied to the study of the thermal distribution [12] for the system. Work in this direction is under investigation.

As a concluding remark we wish to point out that, for some system, q, p and the Hamiltonian H for the system may not constitute a quasi-algebra. In this case, if some other operators and H constitute a quasi-algebra, useful results can be obtained as well by means of the method employed in this paper.

References

[1] Glauber R J 1963 *Phys. Rev.* **131** 2766
[2] Hollenhorst J N 1979 *Phys. Rev.* D **19** 1669
[3] Hartley and Ray J G 1982 *Phys. Rev.* D **25** 382
[4] Lewis H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458

[5] Pedrosa I A 1987 *Phys. Rev.* D **38** 1279
[6] Cheng C M and Fung P C W 1988 *J. Phys. A: Math. Gen.* **21** 4115
[7] Lo C F 1990 *J. Phys. A: Math. Gen.* **23** 1155
[8] Xiao-Chun Gao, Jing-bo Xu and Tie-zheng Qian 1990 *Ann. Phys., NY* **204** 235
[9] Xin Ma 1988 *Phys. Rev.* A **38** 3548
[10] Janszky J and Yushin Y Y 1986 *Opt. Commun.* **59** 151
[11] Graham R 1987 *J. Mod. Opt.* **34** 873
[12] Hamiltonian J 1990 *Phys. Rev.* A **42** 1